

# NEW SOLUTIONS OF THE YANG-BAXTER EQUATION BASED ON ROOT OF 1 REPRESENTATIONS OF THE PARA-BOSE SUPERALGEBRA $U_q[\mathfrak{osp}(1/2)]$

T. D. Palev\* and N. I. Stoilova\*

International Centre for Theoretical Physics, 34100 Trieste, Italy

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**Abstract.** New solutions of the quantum Yang-Baxter equation, depending in general on three arbitrary parameters, are written down. They are based on the root of unity representations of the quantum orthosymplectic superalgebra  $U_q[\mathfrak{osp}(1/2)]$ , which were found recently. Representations of the braid group  $B_N$  are defined within any  $N^{th}$  tensorial power of root of 1  $U_q[\mathfrak{osp}(1/2)]$  modules.

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\* Permanent address: Institute for Nuclear Research and Nuclear Energy, 1784 Sofia, Bulgaria; E-mail: palev@bgearn.bitnet, stoilova@bgearn.bitnet

## 1. Introduction

In the present paper we write down new solutions of the quantum Yang-Baxter equation (QYBE), associated with root of unity representations of the quantum orthosymplectic superalgebra  $U_q[osp(1/2)]$ , which we have recently constructed [1]. All such representations are with a highest and with a lowest weight. For  $q$  being a  $4k$  root of 1 with  $k = 3, 5, 7, \dots$ , there exists a continuous class of  $k$ -dimensional representations. The solutions of the QYBE we find depend in general on three continuous parameters.

The general interest for studying solutions of the quantum Yang-Baxter equation is inspired from the various applications of the latter in conformal field theory [2, 3], quantum integrable models [4, 5] and knot theory [6-8]. Our motivation for the present investigation is of somewhat different nature. It originates from the close connection between the representations of the orthosymplectic superalgebras and the quantum statistics [9, 10], more precisely - the parastatistics [11].

It is perhaps worth commenting the last point in some more details. To this end consider as an example the Hopf algebra  $U_q[osp(1/2n)]$ , the quantized universal enveloping algebra of the orthosymplectic Lie superalgebra  $osp(1/2n)$ . The quantization of the latter in terms of its Chevalley generators is well known [12-17]. Recently there has been given an alternative definition of  $U_q[osp(1/2n)]$  [18-21] in terms of preoscillator generators  $a_i^\pm$ ,  $K_i = q^{H_i}$ ,  $i = 1, \dots, n$ . The relation to the quantum statistics stems from the observation that the operators  $a_i^\pm$ ,  $i = 1, \dots, n$  can be identified with deformed para-Bose operators. Moreover it turns out that the oscillator (or Weyl) superalgebra  $W_q(n)$  generated by  $n$  pairs of deformed Bose operators [22-25] is a factor algebra of  $U_q[osp(1/2n)]$  [26, 27] and (depending on the precise definition of the preoscillator generators) a morphism of  $U_q[osp(1/2n)]$  onto  $W_q(n)$  is given essentially by a replacement of the deformed para-Bose operators with deformed Bose operators. Therefore, despite of the fact that the oscillator algebra  $W_q(n)$  is not a Hopf algebra, one can define an  $R$ -matrix associated with  $W_q(n)$  simply by considering the Fock representation of  $W_q(n)$  also as a representation of  $U_q[osp(1/2n)]$ . To this end one has to express the  $U_q[osp(1/2n)]$  universal  $R$ -matrix in terms of preoscillator generators and subsequently replace them with deformed Bose operators. The related matrices  $R_{12}$ ,  $R_{13}$ ,  $R_{23}$ , which are functions on  $n$  pairs of deformed Bose operators and the corresponding number operators, provide a "bosonic" solution of the QYBE. Certainly one can try to carry out the above programme in a more general framework, considering other representations of the preoscillator generators. This would correspond to finding representations of the deformed para-Bose operators. The problem however is not simple; it has not been solved so far even in the nondeformed case.

The present paper is a small step towards the realization of the above programme. Here we deal with the superalgebra  $U_q[osp(1/2)]$ . Nevertheless already in this simple case one arrives to interesting conclusions. It turns out, for instance, that apart from the representations corresponding to both deformed and nondeformed parabosons (and in particular - bosons) one finds a (root of 1) representation with  $a^\pm$  being usual fermions [28], i.e., the fermions are deformed parabosons. Thus, the bosons and the fermions appear as different irreps of one and the same quantized superalgebra, namely  $U_q[osp(1/2)]$ . As an example we write down the corresponding 4-dimensional (nondiagonal)  $R$ -matrix, which leads to a "fermionic" solution of the QYBE.

The new solutions of the QYBE will be based on the representations of the preoscillator generators  $a^\pm$ ,  $K = q^H$  in (deformed para-Bose) Fock spaces. We pay special attention to the case when the deformation parameter  $q$  is a root of unity, which leads to finite-dimensional Fock spaces.

For  $n > 1$  the preoscillator generators of  $U_q[osp(1/2n)]$  are very different from its Chevalley generators. In case  $n = 1$  however the creation and the annihilation (deformed para-Bose) operators  $a^+$ ,  $a^-$  can be identified with the positive and the negative root vectors  $e$  and  $f$  of  $U_q[osp(1/2)]$ , respectively. Therefore the results to follow could have been given entirely in terms of the canonical for  $U_q[osp(1/2)]$  terminology and notation. We prefer however to keep close to the notation of the preoscillator generators, speaking about creation and annihilation operators instead of Chevalley generators, (deformed) Fock spaces instead of Verma modules, etc. In order to underline that  $U_q[osp(1/2)]$  is (essentially) generated by deformed para-Bose operators we call it a (deformed) para-Bose superalgebra.

The paper is organized as follows. In Sec. 2 we recall the definition of the deformed para-Bose superalgebra and its Fock irreps for  $q$  being root of 1. The form of the transformation relations is new and more compact, comparing to those given in [1]. In Sec. 3 new solutions of the QYBE are constructed. The situation here is rather peculiar. We first prove that at  $q$  being root of 1  $U_q[osp(1/2)]$  is in general not almost cocommutative. Nevertheless the expression of the (generic) universal  $R$ -matrix turns to be well defined within all of our representation spaces, which leads to solutions of the QYBE. In addition the  $R$ -matrix allows us to define representations of the braid group  $B_N$  in the  $N^{th}$  tensorial power of any of the  $U_q[osp(1/2)]$  Fock modules.

Throughout we use the following abbreviations and notation:  $\mathbf{C}$  – all complex numbers;  $\mathbf{Z}$  – all integers;  $\mathbf{Z}_+$  – all nonnegative integers;  $\mathbf{Z}_2 = \{\bar{0}, \bar{1}\}$ ;  $[A, B] = AB - BA$ ,  $\{A, B\} = AB + BA$ ;  $U_q \equiv U_q[osp(1/2)]$ .

## 2. The para-Bose algebra $U_q[osp(1/2)]$ and its Fock irreps

Here we summarize the results of [1]. The form of the expressions (2.5), (2.7)-(2.9), describing the transformations of the Fock spaces, is however new. It is more compact than the corresponding relations in [1].

The superalgebra  $U_q = U_q[osp(1/2)]$ ,  $q \in \mathbf{C} \setminus \{0, \pm 1\}$  has three generators  $a^+$ ,  $a^-$ ,  $H$ , satisfying the defining relations:

$$[H, a^\pm] = \pm 2a^\pm, \quad \{a^+, a^-\} = \frac{q^H - q^{-H}}{q - q^{-1}}. \quad (2.1)$$

$H$  is an even generator,  $a^\pm$  are odd. As  $q \rightarrow 1$   $H = \{a^+, a^-\}$  and eqs. (2.1) reduce to the defining relations of the nondeformed para-Bose operators [11] ( $\xi, \eta, \epsilon = \pm$  or  $\pm 1$ ):

$$[\{a^\xi, a^\eta\}, a^\epsilon] = (\epsilon - \eta)a^\xi + (\epsilon - \xi)a^\eta. \quad (2.2)$$

The Hopf algebra structure on  $U_q$  can be defined in different ways [17]. For the comultiplication we set

$$\Delta(H) = H \otimes 1 + 1 \otimes H, \quad \Delta(a^+) = a^+ \otimes 1 + q^{-H} \otimes a^+, \quad \Delta(a^-) = a^- \otimes q^H + 1 \otimes a^-. \quad (2.3)$$

Passing to the representations of  $U_q$  we note that the finite-dimensional irreps of  $U_q[osp(1/2)]$  at generic  $q$  were constructed in [29, 30]. Some root of unity highest weight irreps were also obtained in [30]; both highest weight and cyclic representations were studied in [31-34].

A (deformed) Fock space  $F(p)$  is defined in the usual for the parastatistics way [11]: for any complex  $p$  (which is an analogue of the order of the parastatistics) one postulates the existence of a vacuum vector

$|0\rangle \in F(p)$  so that  $a^-|0\rangle = 0$  and  $H|0\rangle = p|0\rangle$ . From now on we shall denote by  $a_p^\pm$  and  $H_p$  the representatives of  $a^\pm$  and  $H$  in  $F(p)$ . The latter is an infinite-dimensional linear space with a basis  $|n\rangle = (a_p^+)^n|0\rangle$ ,  $n \in \mathbf{Z}_+$ .

Setting

$$\{n; x\}_q = \frac{q^{n+x} - (-1)^n q^{-n-x}}{q - (-1)^n q^{-1}} \quad (2.4)$$

one can write the transformation of the basis as follows:

$$H_p|n\rangle = (2n+p)|n\rangle, \quad a_p^-|n\rangle = \{n; 0\}_q \{n-1; p\}_q |n-1\rangle, \quad a_p^+|n\rangle = |n+1\rangle. \quad (2.5)$$

At generic  $q$  the space  $F(p)$  is infinite-dimensional. It is a simple (=irreducible)  $U_q$  module if  $p$  is not a negative even number [28] (which we always assume). The space  $F(p=1)$  is the Fock space of deformed Bose operators [22-25]. Within  $F(1)$  the preoscillator operators satisfy the relations

$$a_1^- a_1^+ - q^{\pm 2} a_1^+ a_1^- = q^{\mp 2N}, \quad \text{where } N = \frac{1}{2}(H_1 - 1) \text{ is the number operator.} \quad (2.6)$$

In the root of unity cases  $F(p)$  is indecomposable if and only if  $q = e^{i\frac{\pi}{2}\frac{m}{k}}$  for every  $m, k \in \mathbf{Z}$  such that  $q \notin \{\pm 1, \pm i\}$ . The factor-space of  $F(p)$  with respect to the maximal invariant subspace is an irreducible module, containing the vacuum vector  $|0\rangle$ .

The algebras  $U_q$  corresponding to all possible values of  $m$  and  $k$  contain several isomorphic copies. Without loss of generality we restrict  $m$  and  $k$  to values, which we call admissible, namely: (1)  $k = 2, 3, \dots$ ; (2)  $m \in \{1, 2, \dots, k-1\}$ ; (3)  $m$  and  $k$  are relatively co-prime. From now on we consider  $q = e^{i\frac{\pi}{2}\frac{m}{k}}$  to be only an admissible root of 1.

The irreducible  $U_q$  modules with  $q$  being root of 1 are finite-dimensional. Denote by  $W^L(p) \subset F(p)$  an  $L+1$ -dimensional representation space with a basis  $|0\rangle, |1\rangle, \dots, |L\rangle$ . Its transformations under the action of the  $U_q$  generators read:

$$H_p|n\rangle = (2n+p)|n\rangle, \quad a_p^-|n\rangle = \{n; 0\}_q \{n-1; p\}_q |n-1\rangle, \quad a_p^+|L\rangle = 0, \quad a_p^+|n\rangle = |n+1\rangle, \quad n < L. \quad (2.7)$$

We distinguish two classes of algebras, each one containing three groups of representations,

$$\begin{aligned} \text{Class I } (k-m = \text{odd}) : & \quad (I.a) \ L = 2k-1 \text{ if } p \neq \text{integer}; \quad (I.b) \ L = p(k-1)(\text{mod } 2k) \text{ if } p = \text{integer}; \\ & \quad (I.c) \ L = 2k-1 \end{aligned} \quad (2.8)$$

$$\begin{aligned} \text{Class II } (k, m = \text{odd}) : & \quad (II.a) \ L = k-1 \text{ if } p \neq \text{even}; \quad (II.b) \ L = (k-p)(\text{mod } k) \text{ if } p = \text{even} \\ & \quad (II.c) \ L = k-1. \end{aligned} \quad (2.9)$$

The cases  $(I.a)$ ,  $(I.b)$ ,  $(II.a)$  and  $(II.b)$  correspond to irreducible representations, whereas in  $(I.c)$  (resp.  $(II.c)$ ) the representation is indecomposable if  $p = \text{integer}$  (resp. if  $p = \text{even}$ ). The  $2k$ -dimensional modules corresponding to  $(I.c)$  were described in [32], where in particular it was shown how one can modify those of them corresponding to  $k=\text{odd}$  and  $m=\text{even}$  so that they carry cyclic representations. One has to keep in mind however that at certain values of  $p$  these modules are no more irreducible, they are indecomposable. In fact each simple module  $W^L(p)$  from  $(I.b)$  with  $L = p(k-1)(\text{mod } 2k)$  is a factor-module of  $W^{2k-1}(p)$  from  $(I.c)$  with respect to its maximal invariant subspace. To our best knowledge the representations from the classes  $(I.b)$  and  $II$  were not described in the literature so far.

One can always assume  $0 < \text{Re}(p) \leq 4k$ , since the representations with  $p$  outside that interval are equivalent to representations with  $p$  obeying the above inequality; if  $k$  is odd and  $m$  is even one can further set  $0 < \text{Re}(p) \leq 2k$  if  $m = 2(\text{mod } 4)$  and  $0 < \text{Re}(p) \leq k$  if  $m = 4(\text{mod } 4)$ .

### 3. R-matrices and new solutions of the QYBE

One way for constructing  $R$ -matrices and hence solutions of the QYBE is based on the use of the universal  $R$ -matrix of a quasitriangular Hopf algebra  $U$  together with the representations of  $U$ .

The universal  $R$ -matrix for  $U_q$  was written down in [29, 30]. Here we use the expression as given in [17], which read in our notation:

$$R = \sum_{n \geq 0} (-1)^{\frac{n(n+1)}{2}} \frac{(q - \bar{q})^n}{(n)_{-\bar{q}^2}!} [(a^+)^n \otimes (a^-)^n] q^{\frac{1}{2}H \otimes H} \quad (3.1)$$

where  $\bar{q} = q^{-1}$ ,  $(n)_a = (1 - a^n)/(1 - a)$ ,  $(n)_a! = (1)_a(2)_a \dots (n)_a$ .

If  $\rho_1$  and  $\rho_2$  are two representations of  $U_q$  defined in  $V_1$  and  $V_2$ , then the related  $R$ -matrix is  $R(\rho_1, \rho_2) = (\rho_1 \otimes \rho_2)R \in \text{End}(V_1 \otimes V_2)$ .

In the root of 1 cases however the above construction generally fails, because for certain admissible  $q$   $U_q$  is no more almost cocommutative. The proof is essentially the same as the one given by Arnaudon for  $U_q[\mathfrak{sl}(2)]$  [35]. It is based on the observation that  $U_q$  contains a larger center generated from its Casimir operator and the additional central elements  $\hat{x}^\pm = (a^\pm)^{2k}$  and  $\hat{z} = (K)^{2k}$  [30, 31]. If  $\rho$  is an irrep of  $U_q$  in  $V$ , then  $\rho(\hat{x}^\pm) = \rho(x^\pm)\mathbf{1}_V$ ,  $\rho(\hat{z}) = \rho(z)\mathbf{1}_V$ , where  $\mathbf{1}_V$  is the unit operator in  $V$  and  $\rho(x^\pm), \rho(z) \in \mathbf{C}$ .

We proceed to show that the universal  $R$ -matrix does not exist for a subclass of I, corresponding to all algebras with  $k=\text{odd}$  and  $m=\text{even}$ . To this end we use the following general identity: if  $q$  is an  $l^{\text{th}}$  primitive root of 1 and  $AB + q^2BA = 0$ , then for any  $N \leq l$

$$(A + B)^N = \sum_{n=0}^N q^{-n(N-n)} \left\{ \begin{matrix} N \\ n \end{matrix} \right\}_q A^n B^{N-n}, \quad \left\{ \begin{matrix} N \\ n \end{matrix} \right\}_q = \frac{\{N\}_q!}{\{n\}_q! \{N-n\}_q!}, \quad \{n\}_q = q^n - (-1)^n q^{-n}. \quad (3.2)$$

Applying (3.2) for  $N = 2k$ ,  $A = 1 \otimes a^-$  and  $B = a^- \otimes K$ , we obtain for all class I algebras:

$$\Delta(\hat{x}^-) = 1 \otimes \hat{x}^- + \hat{x}^- \otimes \hat{z}, \quad \Delta^{\text{op}}(\hat{x}^-) = \hat{x}^- \otimes 1 + \hat{z} \otimes \hat{x}^-. \quad (3.3)$$

In (3.3)  $\Delta^{\text{op}} = \sigma\Delta$  is the opposite comultiplication;  $\sigma$  is a superpermutation,  $\sigma(a \otimes b) = (-1)^{\deg(a)\deg(b)} b \otimes a$ . Assume now that  $U_q$  is almost cocommutative, namely that there exists an invertible element  $R$  from (the completion of)  $U_q \otimes U_q$ , so that  $R\Delta(a) = \Delta^{\text{op}}(a)R$  for any  $a \in U_q$ . On the tensor product of two irreps  $\rho_1$  and  $\rho_2$  in  $V_1$  and  $V_2$  one would have for  $a = \hat{x}^-$ :

$$(\rho_1 \otimes \rho_2)(R)(\rho_1 \otimes \rho_2)(\Delta(\hat{x}^-)) = (\rho_1 \otimes \rho_2)(\Delta^{\text{op}}(\hat{x}^-))(\rho_1 \otimes \rho_2)(R). \quad (3.4)$$

Acting with both sides of (3.4) on an arbitrary vector  $|X\rangle \in V_1 \otimes V_2$ , one gets:

$$\{\rho_1(x^-) + \rho_1(z)\rho_2(x^-)\}|Y\rangle = \{\rho_2(x^-) + \rho_2(z)\rho_1(x^-)\}|Y\rangle,$$

where  $|Y\rangle = (\rho_1 \otimes \rho_2)(R)|X\rangle$ . Therefore

$$\rho_2(x^-) + \rho_2(z)\rho_1(x^-) = \rho_1(x^-) + \rho_1(z)\rho_2(x^-). \quad (3.5)$$

The central elements  $\hat{x}^\pm$  and  $\hat{z}$  can take arbitrary values on the cyclic irreps of the algebras with  $k=\text{odd}$  and  $m=\text{even}$  [34], i.e., in this case  $\rho_1(x^-)$ ,  $\rho_2(x^-)$ ,  $\rho_1(z)$  and  $\rho_2(z)$  are arbitrary numbers, which is in contradiction to (3.5). Therefore the universal  $R$ -matrix cannot exist for these algebras. Note however that (3.5) is not in contradiction with the representations (2.8), since for any of them  $\rho(x^\pm) = 0$ . Therefore, following Rosso [36], one can try to produce an almost universal  $R$ -matrix on the quotient  $\tilde{U}_q = U_q[\mathfrak{osp}(1/2)]/(\hat{x}^\pm = 0)$ .

Eq. (3.5) holds for the subclass of the class I algebras, corresponding to  $k=\text{even}$  and  $m=\text{odd}$ . The known irreps for this subclass are only those listed in (2.8). The latter do not contradict (3.5), since  $\hat{x}^\pm$  and  $\hat{z}$  act as zero operators within each class I  $U_q$ -module. Therefore the question whether the universal  $R$  matrix exists for the algebras with  $k=\text{even}$  and  $m=\text{odd}$  is an open one. The same holds for all algebras from the class II. Within each  $U_q$  module corresponding to (2.9)  $\hat{x}^\pm$  and  $\hat{z}$  are zero operators. Our attempts to extend these modules to carry cyclic representations were not successful. Moreover, the eqs. (3.3) and hence (3.5) are no more true.

We see that the question about the existence of an universal  $R$  matrix for the algebras  $U_q$  with  $q$  being root of 1 cannot be answered uniquely at present. Our claim is that the  $R$ -matrix (3.1), considered as an element from  $\tilde{U}_q \otimes \tilde{U}_q$  is almost universal, namely it is "universal" for all Fock representations (2.7)-(2.9): if  $\rho^{L_1}(p_1)$  and  $\rho^{L_2}(p_2)$  are any two such representations, then the operator

$$R^{L_1, L_2}(p_1, p_2) = (\rho^{L_1}(p_1) \otimes \rho^{L_2}(p_2))(R) : W^{L_1}(p_1) \otimes W^{L_2}(p_2) \rightarrow W^{L_1}(p_1) \otimes W^{L_2}(p_2) \quad (3.6)$$

satisfies the analogue of (3.4),

$$R^{L_1, L_2}(p_1, p_2)(\rho^{L_1}(p_1) \otimes \rho^{L_2}(p_2))(\Delta(a)) = (\rho^{L_1}(p_1) \otimes \rho^{L_2}(p_2))(\Delta^{op}(a))R^{L_1, L_2}(p_1, p_2). \quad (3.7)$$

The explicit action of  $R^{L_1, L_2}(p_1, p_2)$  on the basis  $|l_1\rangle \otimes |l_2\rangle$  of  $W^{L_1}(p_1) \otimes W^{L_2}(p_2)$  yields:

$$\begin{aligned} R^{L_1, L_2}(p_1, p_2)(|l_1\rangle \otimes |l_2\rangle) &= q^{\frac{1}{2}(2l_1+p_1)(2l_2+p_2)} \sum_{n=0}^{\min(L_1-l_1, l_2)} (-1)^{\frac{n}{2}(n+2l_1+1)} \frac{(q-\bar{q})^n}{(n)_{-\bar{q}^2}!} \\ &\times \prod_{i=0}^{n-1} \{l_2 - i; 0\}_q \{l_2 - 1 - i; p_2\}_q |l_1 + n\rangle \otimes |l_2 - n\rangle. \end{aligned} \quad (3.8)$$

The proof of (3.7) is by a direct computation within each  $U_q$  module  $W^{L_1}(p_1) \otimes W^{L_2}(p_2)$ , i.e., using the transformation relations (3.8).

The linear operators

$$R_{12}^{L_1, L_2}(p_1, p_2), R_{13}^{L_1, L_3}(p_1, p_3), R_{23}^{L_2, L_3}(p_2, p_3) \text{ in } W^{L_1, L_2, L_3}(p_1, p_2, p_3) \equiv W^{L_1}(p_1) \otimes W^{L_2}(p_2) \otimes W^{L_3}(p_3) \quad (3.9)$$

which satisfy the QYBE

$$R_{12}^{L_1, L_2}(p_1, p_2) R_{13}^{L_1, L_3}(p_1, p_3) R_{23}^{L_2, L_3}(p_2, p_3) = R_{23}^{L_2, L_3}(p_2, p_3) R_{13}^{L_1, L_3}(p_1, p_3) R_{12}^{L_1, L_2}(p_1, p_2). \quad (3.10)$$

are defined on the basis as follows:

$$\begin{aligned}
R_{12}^{L_1, L_2}(p_1, p_2)(|l_1\rangle \otimes |l_2\rangle \otimes |l_3\rangle) &= q^{\frac{1}{2}(2l_1+p_1)(2l_2+p_2)} \sum_{n=0}^{\min(L_1-l_1, l_2)} (-1)^{\frac{n}{2}(n+2l_1+1)} \\
&\times \frac{(q-\bar{q})^n}{(n)_{-\bar{q}^2}!} \prod_{i=0}^{n-1} \{l_2-i; 0\}_q \{l_2-1-i; p_2\}_q |l_1+n\rangle \otimes |l_2-n\rangle \otimes |l_3\rangle.
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
R_{13}^{L_1, L_3}(p_1, p_3)(|l_1\rangle \otimes |l_2\rangle \otimes |l_3\rangle) &= q^{\frac{1}{2}(2l_1+p_1)(2l_3+p_3)} \sum_{n=0}^{\min(L_1-l_1, l_3)} (-1)^{\frac{n}{2}(n+2l_1+2l_2+1)} \\
&\times \frac{(q-\bar{q})^n}{(n)_{-\bar{q}^2}!} \prod_{i=0}^{n-1} \{l_3-i; 0\}_q \{l_3-i-1; p_3\}_q |l_1+n\rangle \otimes |l_2\rangle \otimes |l_3-n\rangle.
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
R_{23}^{L_2, L_3}(p_2, p_3)(|l_1\rangle \otimes |l_2\rangle \otimes |l_3\rangle) &= q^{\frac{1}{2}(2l_2+p_2)(2l_3+p_3)} \sum_{n=0}^{\min(L_2-l_2, l_3)} (-1)^{\frac{n}{2}(n+2l_2+1)} \\
&\times \frac{(q-\bar{q})^n}{(n)_{-\bar{q}^2}!} \prod_{i=0}^{n-1} \{l_3-i; 0\}_q \{l_3-i-1; p_3\}_q |l_1\rangle \otimes |l_2+n\rangle \otimes |l_3-n\rangle.
\end{aligned} \tag{3.13}$$

The operators (3.9) can be expressed in terms of the  $R$ -matrix (3.8). To this end introduce a superpermutation linear operator  $P_{23} : (|n_1\rangle \otimes |n_2\rangle) \otimes |n_3\rangle = (-1)^{n_2 n_3} |n_1\rangle \otimes |n_3\rangle \otimes |n_2\rangle$ . Then

$$\begin{aligned}
R_{12}^{L_1, L_2}(p_1, p_2) &= R^{L_1, L_2}(p_1, p_2) \otimes 1, \\
R_{13}^{L_1, L_3}(p_1, p_3) &= P_{23}(R^{L_1, L_3}(p_1, p_3) \otimes 1)P_{23}, \\
R_{23}^{L_2, L_3}(p_2, p_3) &= 1 \otimes R^{L_2, L_3}(p_2, p_3).
\end{aligned} \tag{3.14}$$

Depending on the choice of the representations (2.8) and (2.9) one obtains  $R$ -matrices of different dimensions, which may be parameter independent or can depend on one or two free parameters.

If  $\rho^{L_1}(p_1), \rho^{L_2}(p_2) \in (I.c)$ , then  $R^{L_1, L_2}(p_1, p_2)$  depends on two arbitrary complex parameters  $p_1$  and  $p_2$ ,  $\dim(R^{L_1, L_2}(p_1, p_2)) = 4k^2$ . These  $R$ -matrices were obtained in [32]. The expression (3.8) is somewhat more compact.

If  $\rho^{L_1}(p_1), \rho^{L_2}(p_2) \in (II.c)$   $R^{L_1, L_2}(p_1, p_2)$  depends also on the arbitrary complex parameters  $p_1$  and  $p_2$ , but  $\dim(R^{L_1, L_2}(p_1, p_2)) = k^2$ . This is a new class of  $R$ -matrices, leading through (3.14) to new solutions of the QYBE, defined in a  $k^3$ -dimensional space  $W^{L_1, L_2, L_3}(p_1, p_2, p_3)$  with  $k = 3, 5, 7, \dots$  and depending on three arbitrary parameters.

In all other cases the  $R$ -matrices depend on less than two free parameters, which is due to the case that for certain values of  $p_1, p_2$  and  $p_3$   $W^{L_1, L_2, L_3}(p_1, p_2, p_3)$  contains invariant subspaces. Those, corresponding to  $\rho^{L_1}(p_1), \rho^{L_2}(p_2) \in (I.b)$  or  $(II.b)$  lead to constant  $R$ -matrices and hence to constant solutions of the QYBE. Here are some examples.

*Example 1.* The representation  $(I.b)$  with  $k = 2$ ,  $(m = 1)$  and  $p = 1$  gives  $L = 1$ . From (2.7) one concludes that  $a^\pm$  are Fermi operators. In the basis  $\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$  the "fermionic"  $R$ -matrix

reads:

$$R^{L_1=1, L_2=1}(p_1=1, p_2=1) = \begin{pmatrix} e^{\frac{i\pi}{8}} & 0 & 0 & 0 \\ 0 & e^{\frac{3i\pi}{8}} & 0 & 0 \\ 0 & e^{\frac{i\pi}{8}} - e^{\frac{5i\pi}{8}} & e^{\frac{3i\pi}{8}} & 0 \\ 0 & 0 & 0 & -e^{\frac{i\pi}{8}} \end{pmatrix}. \quad (3.15)$$

It contains no free parameters.

*Example 2.* We consider the class II algebra  $U_q$  with the smallest possible value of  $k$ , namely  $k=3$  (and hence  $m=1$ ), i.e.,  $q=e^{i\pi/6}$ . There is a tree of  $R$ -matrices, related to the different possible branches of the representations (II.a,b,c). One such branch is, for instance,  $R^{2,2}(p_1, p_2) \rightarrow R^{2,1}(p_1, 2) \rightarrow R^{1,1}(2, 2)$ . The root  $R$ -matrix  $R^{L_1=2, L_2=2}(p_1, p_2)$  is 9-dimensional and depends on two arbitrary parameters  $p_1$  and  $p_2$ . In a matrix form ( ordering the basis lexically,  $|i\rangle \otimes |j\rangle < |k\rangle \otimes |l\rangle$  if  $i < k$  or if  $i = k$  and  $j < l$ ) one obtains from (3.8):

$$R^{2,2}(p_1, p_2) = \begin{pmatrix} A_{00,00} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{01,01} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{02,02} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_{10,01} & 0 & A_{10,10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{11,02} & 0 & A_{11,11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{12,12} & 0 & 0 & 0 \\ 0 & 0 & A_{20,02} & 0 & A_{20,11} & 0 & A_{20,20} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{21,12} & 0 & A_{21,21} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{22,22} \end{pmatrix}. \quad (3.16)$$

with

$$A_{00,00} = e^{\frac{i\pi}{12}p_1p_2} \quad A_{01,01} = e^{\frac{i\pi}{12}p_1(p_2+2)} \quad A_{02,02} = e^{\frac{i\pi}{12}p_1(p_2+4)} \quad A_{10,10} = e^{\frac{i\pi}{12}(p_1+2)p_2} \quad A_{11,11} = e^{\frac{i\pi}{12}(p_1+2)(p_2+2)}$$

$$A_{12,12} = e^{\frac{i\pi}{12}(p_1+2)(p_2+4)} \quad A_{20,20} = e^{\frac{i\pi}{12}(p_1+4)p_2} \quad A_{21,21} = e^{\frac{i\pi}{12}(p_1+4)(p_2+2)} \quad A_{22,22} = e^{\frac{i\pi}{12}(p_1+4)(p_2+4)}$$

$$A_{10,01} = -2ie^{\frac{i\pi}{12}p_1(p_2+2)} \sin\left(\frac{\pi}{6}p_2\right) \quad A_{11,02} = -2ie^{\frac{i\pi}{12}p_1(p_2+4)} \cos\left(\frac{\pi}{6}(p_2+1)\right)$$

$$A_{20,11} = 2ie^{\frac{i\pi}{12}(p_1+2)(p_2+2)} \sin\left(\frac{\pi}{6}p_2\right) \quad A_{20,02} = -ie^{\frac{i\pi}{12}(p_1(p_2+4)+2)} (2\sin\left(\frac{\pi}{6}(2p_2+1)\right) - 1)$$

$$A_{21,12} = 2ie^{\frac{i\pi}{12}(p_1+2)(p_2+4)} \cos\left(\frac{\pi}{6}(p_2+1)\right).$$

Setting  $p_2=2$  and  $L_2=1$  one obtains the next matrix from the branch, namely the 6-dimensional  $R$ -matrix  $R^{L_1=2, L_2=1}(p_1, p_2=2)$ , which depends on the arbitrary parameter  $p_1$ :

$$R^{2,1}(p_1, 2) = \begin{pmatrix} e^{\frac{i\pi}{6}p_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\frac{i\pi}{3}p_1} & 0 & 0 & 0 & 0 \\ 0 & -i\sqrt{3}e^{\frac{i\pi}{3}p_1} & e^{\frac{i\pi}{6}(p_1+2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{i\pi}{3}(p_1+2)} & 0 & 0 \\ 0 & 0 & 0 & i\sqrt{3}e^{\frac{i\pi}{3}(p_1+2)} & e^{\frac{i\pi}{6}(p_1+4)} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\frac{i\pi}{3}(p_1+4)} \end{pmatrix}. \quad (3.17)$$

$R^{2,1}(p_1, 2)$  can be obtained from the root matrix (3.16) by crossing out its rows and columns with numbers 3, 6 and 9 and setting  $p_2=2$ .



The last matrix from the branch corresponds to  $p_1 = p_2 = 2$  and  $L_1 = L_2 = 1$ . It is a 4-dimensional constant  $R$ -matrix, which can be obtained by crossing out the last two rows and columns in (3.17) and setting  $p_1 = 2$ :

$$R^{1,1}(2, 2) = \begin{pmatrix} e^{\frac{i\pi}{3}} & 0 & 0 & 0 \\ 0 & e^{\frac{2i\pi}{3}} & 0 & 0 \\ 0 & -i\sqrt{3}e^{\frac{2i\pi}{3}} & e^{\frac{2i\pi}{3}} & 0 \\ 0 & 0 & 0 & -e^{\frac{i\pi}{3}} \end{pmatrix}. \quad (3.18)$$

One can choose certainly other branches from the  $R$ -matrix tree, obtaining in this way new  $R$ -matrices of smaller dimensions, which are always submatrices of the root matrix (3.16).

Let us mention at the end, following Zhang [38], that the  $R$ -matrix can be used also in order to define representations of the braid group  $B_N$  acting in any  $N^{th}$  tensorial power of Fock spaces  $W^L(p)$ , namely in  $W^L(p)^{\otimes N}$ . To this end set  $\check{R}^L(p) = PR^{L,L}(p, p) \in \text{End}(W^L(p) \otimes W^L(p))$ , where  $P$  is the superpermutation operator in  $W^L(p) \otimes W^L(p)$ . It is straightforward to verify that  $\check{R}^L(p)$  is an  $U_q[\text{osp}(1/2)]$  intertwining operator in  $W^L(p) \otimes W^L(p)$ :

$$[\check{R}^L(p), \Delta(a)] = 0 \quad \forall a \in U_q. \quad (3.19)$$

Hence [38]  $\sigma_i \in \text{End}(W^L(p)^{\otimes N})$   $i = 1, \dots, N-1$ , defined as

$$\sigma_i = \mathbf{1}^{\otimes(i-1)} \otimes \check{R}^L(p) \otimes \mathbf{1}^{\otimes(N-i-1)} \quad (3.20)$$

gives a representation of  $B_N$ , namely the  $\sigma_1, \dots, \sigma_{N-1}$  satisfy the defining relations for  $B_N$ :

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i - j| > 1; \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \quad (3.21)$$

Hence (the representation of the braid group)  $B_N$  is a subset of the set of all intertwining operators in  $W^L(p)^{\otimes N}$ .

#### 4. Concluding remarks

We have found new solutions of the quantum Yang-Baxter equations, using essentially the representations of  $U_q[\text{osp}(1/2)]$ , which we have recently constructed. The solutions were obtained formally from the "generic"  $R$ -matrix (3.1), despite of the fact that the latter does not exist in root of 1 cases. The more precise statement is that at values of the deformation parameter  $q = e^{i\frac{\pi}{2}\frac{m}{k}}$  with  $k = \text{odd}$  and  $m = \text{even}$   $U_q[\text{osp}(1/2)]$  is not quasitriangular and even less - it is not almost cocommutative. In all other admissible cases the question about the existence of  $R$  is an open one.

The results we have announced in the present paper are more of a mathematical nature. The very fact however that  $a^\pm$  are deformed para-Bose operators (in some other terminology - deformed supersingletons [39]) indicates already their relation to quantum physics. In fact the representation with  $p = 1$  corresponds deformed to bosons [22-25]. The one-dimensional quantum oscillator based on such operators exhibits quite unusual properties at  $q$  being root of 1. In particular it leads to discretization of the spectrum of the position and the momentum operators, thus putting the phase space on a lattice [40]. It will be interesting to consider the same problem in the frame of the more general para-Bose oscillator, considering all its (unitarizable) root of 1 representations.

Various kinds of oscillators based on deformed parabosons were discussed in the literature so far (see [1] for references in this respect) without usually paying attention to the underlying coalgebra structure. The arbitrary deformations may face however serious problems: if the underlying deformed para-Bose algebra is not a Hopf algebra (or at least an associative algebra with a comultiplication, which is an algebra morphism), it is impossible to define tensor products of representations. The deformations of the parabosons we consider are free of this disadvantage, since our deformed algebra is identical with the Hopf algebra  $U_q[osp(1/2)]$ . Another positive feature of the Hopf algebra deformations is the existence of an  $R$ -matrix within every Fock space  $W^L(p)$ . The latter allows one to define an action of the braid group  $B_N$  within any  $N^{th}$  tensorial power  $W^L(p)^{\otimes N}$ , which commutes with  $U_q[osp(1/2)]$ . This is a step towards the decomposition of  $W^L(p)^{\otimes N}$  into irreducible  $U_q[osp(1/2)]$  modules.

It will be interesting to generalize the present approach to the case of several, say,  $n$  modes of preoscillator operators. To this end one has first to express the universal  $U_q[osp(1/2n)]$   $R$ -matrix in terms of deformed para-Bose operators and then consider root of 1 representations of them. A good candidate for such a representation is the one of the  $q$ -commuting deformed Bose operators, introduced recently in [20, 21], which permit only root of 1 (unitary) representations.

## Acknowledgments

All results of the present investigation were obtained during our three months visit at the International Centre for Theoretical Physics in Trieste. We are grateful to Prof. Randjbar-Daemi for the invitation and for the kind hospitality at the High Energy Section of ICTP. We are thankful to Prof. D. Arnaudon for the constructive remarks and suggestions. The work was supported by the Grant  $\Phi - 416$  of the Bulgarian Foundation for Scientific Research.

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